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Extended conformal symmetries and U(1) currents

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Abstract. We investigate the structure of extended conformal algebras generated by local chiral fields of spin $1, 2, \ldots, n$ using the Gelfand-Dickey algebra of formal pseudodifferential operators for the gl(n) algebra. The commutation relations of Zamolodchikov's spin-3 operator algebra are derived in the presence of an additional U(1) conserved current.

1. Introduction

In a remarkable paper [1], Zamolodchikov initiated the study of extended conformal symmetries generated by chiral conformal fields of spin $s \geq 2$, i.e. local fields of weight $(\Delta, \bar{\Delta}) = (s, 0)$. Furthermore, the representation theory of higher-spin operator algebras was developed and applied to various model of two-dimensional conformal field theory that exhibit discrete Z_n symmetry (see for instance [2]). More recently, an alternative (Hamiltonian) description of two-dimensional extended conformal algebras was obtained using the rich algebraic structures of the classical inverse scattering method and the related hierarchies of integrable non-linear differential equations [3]. In particular, the spin-n operator algebra W_n generated by the stress-energy tensor T and the collection of the spin-s primary fields $\{w_s, s = 3, 4, \ldots, n\}$ that might arise in a given theory was described in terms of the commutation relations of the Gelfand-Dickey algebra of formal pseudodifferential operators, GD(sl(n)), associated with the Lie algebra sl(n) (see also [4] for related results). In this framework, the simplest example ($n = 2$) is provided by the Virasoro algebra alone

$$[T(z), T(z')] = (T(z) + T(z'))\partial_z\delta(z - z') + \frac{c}{12}\partial_z^3\delta(z - z') \qquad (1)$$

with central charge c.

In what follows, we first study the transformation properties of Lax operators under arbitary reparametrisations of the circle. This is useful in identifying the various conformal fields. We then give a brief exposition of the theory of Gelfand-Dickey algebras with emphasis placed on the general case GD(gl(n)). This way, we are able to incorporate in the formalism the presence of additional U(1) conserved currents that arise in two-dimensional field theories. Extended conformal algebras that also

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contain spin-1 fields are obtained and their explicit form is presented for $n=2,3$. We note that the restriction to $\mathrm{sl}(n)\subset \mathrm{gl}(n)$ amounts to setting the U(1) fields zero, thus reducing the operator algebras we construct to the spin-n algebras W_n. Finally, we comment on the possiblity of introducing spin-1 fields associated with non-Abelian Kac-Moody symmetries in extended conformal algebras using a suitable generalisation of the Gelfand-Dickey framework.

2. Transformations of Lax operators

Let us consider first the group of (orientation preserving) diffeomorphisms of the circle S^1 and its representation on the space of linear differential operators of fixed degree n:

$$L_n=\partial^n+u_{n-1}(z)\partial^{n-1}+\cdots+u_1(z)\partial+u_0(z). \tag{2}$$

It is obvious that under arbitary reparametrisations $z\to\sigma(z)$, $\sigma\in \mathrm{Diff}\, S^1$, the differential operators (2) transform into each other by changing the configuration of the 'coordinate' functions $u_{n-1},\ldots,u_0$ accordingly. For the purposes of conformal field theory it is most convenient to think of the operators L_n as acting on densities of weight $-(n-1)/2$ rather than on scalar functions. Notice that in this case, the choice $u_{n-1}=0$ remains unchanged under arbitary reparametrisations, i.e. the space of differential operators (2) forms an invariant subspace under the action of Diff S^1 (see for instance [3-5]).

With this in mind, we work out the transformation of the functions $u_{n-1},\ldots,u_0$ under $z\to\sigma(z)$. We find that

$$L_n\to\sigma'^{-(n+1)/2\sigma}L_n\sigma'^{-(n-1)/2} \tag{3}$$

where ${}^\sigma L_n=\partial^n+{}^\sigma u_{n-1}(z)\partial^{n-1}+\cdots+{}^\sigma u_1(z)\partial+{}^\sigma u_0(z)$. The explicit form of ${}^\sigma u_{n-1},\ldots,{}^\sigma u_0$ is quite complicated and it will not be given in the general case. However, it is fairly straightforward to check that for the functions u_{n-1} and u_{n-2} the following is true:

$${}^\sigma u_{n-1}(z)=\sigma' u_{n-1}(\sigma(z)) \tag{4}$$

$${}^\sigma u_{n-2}(z)=\sigma'^2u_{n-2}(\sigma(z))+\frac{n-1}{2}\,\sigma'' u_{n-1}(\sigma(z))+\frac{n^3-n}{12}\,S_\sigma(z). \tag{5}$$

Here, $S_\sigma(z)$ is the Schwartzian derivative of σ, i.e. $S_\sigma(z)=(\sigma'''/\sigma')-\frac{3}{2}(\sigma''/\sigma')^2$. At this point notice that u_{n-1} transforms like a primary conformal field of weight 1, as implied by equation (4). Also, the linear combination

$$w_2(z)=u_{n-2}(z)-\frac{n-1}{2}\,u'_{n-1}(z) \tag{6}$$

transforms (up to a Schwartzian term) like a quadratic differential. Indeed

$${}^\sigma w_2(z)=\sigma'^2w_2(\sigma(z))+\frac{n^3-n}{12}\,S_\sigma(z) \tag{7}$$

and so w_2 behaves like the stress-energy tensor of two-dimnsional conformal field theories. We note that the one-parameter family of fields $w_2^a(z):=u_{n-2}(z)-[(n-1)/2]u'_{n-1}(z)+au_{n-1}^2$ also obeys the transformation law (7) for all values of the numerical constant a. For reasons that will become clear later, we choose to work with $a=-\frac{1}{2}$ and so we define

$$T(z):=u_{n-2}(z)-\frac{n-1}{2}\,u'_{n-1}(z)-\tfrac{1}{2}u_{n-1}^2(z). \tag{8}$$

Before proceeding further a remark is in order. The space of differential operators $\partial^2 + u_0(z)$ transforms according to (3) for $n=2$ and $u_1 = 0$. In this case, equation (5) (or equivalently (7)) identifies the space of Hill–Schrödinger operators $\partial^2 + u_0(z)$ with the centrally extended space of quadratic differentials. The latter is isomorphic with the smooth dual of the Virasoro algebra (1) and so (3) describes the coadjoint representation of the Virasoro algebra (see [6] for more details and further applications). For $n>2$, equation (3) provides more general representations of Diff S^1 that will play an important role in our treatment.

The transformation laws (4), (5) suggest that the functions u_{n-j} $(j=1,2,\ldots,n)$ are associated with conformal fields of weight j. For instance, for $n=3$, explicit calculation yields

$$^{\sigma}u_0(z) = \sigma'^3 u_0(\sigma(z)) + \sigma'\sigma'' u_1(\sigma(z)) + \sigma''' u_2(\sigma(z)) - \frac{\sigma''^2}{\sigma'} u_2(\sigma(z)) + S'_\sigma(z). \tag{9}$$

Although $u_0(z)$ does not transform as a primary conformal field of spin 3, the combination

$$w_3^b := u_0 - \tfrac{1}{2}u_1' + \tfrac{1}{6}u_2'' - \tfrac{1}{3}u_1 u_2 + \tfrac{1}{3}u_2 u_2' + b u_2^3 \tag{10}$$

does so, for all values of the parameter b. More generally, it is possible to find $^{\sigma}u_{n-j}(z)$ for all fields entering in equation (3) and then consider appropriate combinations of them to construct conformal fields of weight 3, 4, ..., n. Primary fields of integer conformal weight $j \leqslant n$ are of the general form

$$w_j(z) = \sum_{\{i\},\{k\}} C^{k_1 \ldots k_p}_{n;i_1 \ldots i_p} u^{(k_1)}_{n-i_1}(z) \ldots u^{(k_p)}_{n-i_p}(z) \tag{11}$$

where $\{i\}$, $\{k\}$ are sets of integers ($\geqslant 0$) that satisfy the condition $k_1 + \cdots + k_p + i_1 + \cdots + i_p = j$. (Here $u^{(k)}(z) := \mathrm{d}^k u(z)/\mathrm{d}z^k$.) The determination of the numerical constants $C^{\{k\}}_{n;\{i\}}$ requires lengthy computations when n is large; here we restrict ourselves to the cases $n=2$, 3 (see also [7]). We note that for the Diff S^1 invariant subspace of differential operators L_n with $u_{n-1}=0$, the expressions given above reduce to those derived in [3].

3. Conformal algebras

Having investigated the behaviour of $u_{n-j}(z)$ $(1 \leqslant j \leqslant n)$ under arbitary reparametrisations of the circle we may proceed further and obtain the commutation relations of their algebra. However, the present formalism is only suggestive of the content of the operator algebra generated by the chiral conformal fields $w_j(z)$ of integer spin $j = 1, 2, \ldots, n$. The appropriate algebraic structure for this purpose is provided by the Gelfand–Dickey algebra. This is defined in terms of the operators $L_n = \partial^n + u_{n-1}(z)\partial^{n-1} + \cdots + u_0(z)$ as follows: let X_f denote the formal sum

$$X_f = \sum_{i=1}^{n} \partial^{-i} x_i \qquad x_i = \frac{\delta f}{\delta u_{i-1}} \tag{12}$$

for the functionals $f = f[u_0, u_1, \ldots, u_{n-1}]$. Then for any two functionals f, g we introduce the bracket [8]

$$\{f, g\}_n = \int \mathrm{res}(V_{X_f}(L_n) X_g) \tag{13}$$

where

$$V_{X_f}(L_n) = L_n(X_f L_n)_+ - (L_n X_f)_+ L_n \tag{14}$$

with the product of operators defined using the Leibnitz rule. We also use the notation $A_+ = \Sigma_{i \geqslant 0} a_i \partial^i$ and res $A = a_{-1}$ for any $A = \Sigma_i a_i \partial^i$. The bracket $\{ , \}_n$ is antisymmetric and satisfies the Jacobi identity for all functionals of $u_0, u_1, \ldots, u_{n-1}$. The algebra constructed this way is called the Gelfand-Dickey algebra GD(gl(n)) [5, 8]. We mention briefly that the notation adopted here is connected with the fact that the space of differential operators L_n can be embedded in the smooth dual of the gl(n) Kac-Moody algebra (for further details see [9]).

In appendices 1 and 2, we write down the explicit form of the GD(gl(n)) bracket for $n = 2$ and $n = 3$ respectively. The formulae become much more complicated for $n \geqslant 4$ and we do not write them down. In all cases, the result follows by straightforward algebraic manipulations of equation (13).

Next we apply the framework of Gelfand-Dickey algebras to construct operator algebras of two-dimensional conformal field theory. In general, the (primary) conformal fields $w_j(z)$ $j = 1, 2, \ldots, n$ are local functionals of the 'coordinate' functions $u_0, u_1, \ldots, u_{n-1}$ (cf (11)) and their bracket can be calculated using (13). For $n > 2$ the resulting operator algebras are not Lie algebras because their determining relations are quadratic (and higher) in the generators w_j. However, it is easy to verify that $w_1 := u_{n-1}$ generates a U(1) Kac-Moody algebra while the spin-2 field $T(z)$ given by (8) satisfies the Virasoro algebra (1). We find that

$$\{u_{n-1}(z),\ u_{n-1}(z')\}_n = -n\partial_z\delta(z-z') \tag{15}$$

$$\{T(z),\ u_{n-1}(z')\}_n = u_{n-1}(z)\partial_z\delta(z-z') \tag{16}$$

$$\{T(z),\ T(z')\}_n = (T(z)+T(z'))\partial_z\delta(z-z') + \frac{n^3-n}{12}\partial_z^3\delta(z-z'). \tag{17}$$

The commutation relations (15)-(17) describe the combined U(1) Kac-Moody and Virasoro symmetry algebra which plays an important role in string theory (see [10] and references therein). We note that the central charge of the Virasoro algebra is $c = n^3 - n$. This follows from the construction of the bracket $\{ , \}_n$ for the particular operator $L_n = \partial^n + u_{n-1}\partial^{n-1} + \cdots + u_0$. Using instead the operator $\lambda\partial^n + u_{n-1}\partial^{n-1} + \cdots + u_0$ with $\lambda = c/(n^3 - n)$ all values of the central charge c can be obtained.

From the general form of the Gelfand-Dickey bracket (13) we find that the quadratic differentials of the form $w_2^a(z) = u_{n-2}(z) - [(n-1)/2]u'_{n-1}(z) + au_{n-1}^2$ satisfy the following commutation relations:

$$\{w_2^a(z), u_{n-1}(z')\}_n = \gamma_1 u_{n-1}(z)\partial_z\delta(z-z') \tag{18}$$

$$\begin{aligned}\{w_2^a(z),\ w_2^a(z')\}_n &= (w_2^a(z)+w_2^a(z'))\partial_z\delta(z-z') + \frac{n^3-n}{12}\partial_z^3\delta(z-z') \\ &\quad + \gamma_2(u_{n-1}^2(z) + u_{n-1}^2(z'))\partial_z\delta(z-z').\end{aligned} \tag{19}$$

The numerical coefficients γ_1, γ_2 are given by

$$\gamma_1 = -2an - n + 1 \qquad \gamma_2 = 2na^2 + (2n-1)a + \frac{n-1}{2} \tag{20}$$

and so the combined U(1) Kac-Moody and Virasoro algebra emerges when $\gamma_1 - 1 = 0 = \gamma_2$. These conditions are satisfied by $a = -\frac{1}{2}$ for all n. (For the cases $n = 2, 3$ equations (18) and (19) can be verified explicitly using the accompanying appendices.) This justifies the choice (8) for the stress-energy tensor $T(z)$ that we made earlier.

It is quite clear now that the space of differential operators $L_1 = \partial + u_0(z)$ is associated with the U(1) Kac-Moody algebra while the operators $L_2 = \partial^2 + u_1(z)\partial + u_0(z)$ are associated with the combined U(1) Kac-Moody and Virasoro conformal symmetry. Moreover, the use of the Gelfand-Dickey algebras GD(gl(n)) for $n \geq 3$ leads to higher-spin algebras associated with the space of differential operators $L_n = \partial^n + u_{n-1}\partial^{n-1} + \cdots + u_0$. These algebras contain both the Virasoro and U(1) Kac-Moody symmetries as subalgebras (cf (15)-(17)) as well as other primary fields of spin ≥ 3. For $n = 3$ the conformal fields $w_1(z) := u_2(z)$, $T(z) = u_1(z) - u_2'(z) - \frac{1}{2}u_2^2(z)$ and w_3^b of equation (10) generate the spin-3 operator algebra in the presence of a (conserved) U(1) current. Indeed, making use of the bracket $\{\,,\}_n$ given in appendix 2 we find that

$$\{T(z),\ w_3^b(z')\}_3 = (w_3^b(z) + 2w_3^b(z'))\partial_z\delta(z-z') \tag{21}$$

$$\{w_3^b(z),\ w_1(z')\}_3 = (\tfrac{2}{3} - 9b)w_1^2(z)\partial_z\delta(z-z'). \tag{22}$$

From (22) we note that the spin-1 and spin-3 fields can be taken to commute for the value of the parameter $b = \frac{2}{27}$. From now on we call $w_3(z) := w_3^{b=2/27}$ and we obtain the following result:

$$\begin{aligned}\{w_3(z),\ w_3(z')\}_3 = &-\tfrac{1}{6}\partial_z^5\delta(z-z') - \tfrac{5}{12}(T(z) + T(z'))\partial_z^3\delta(z-z')\\ &+\tfrac{1}{4}(T''(z) + T''(z'))\partial_z\delta(z-z')\\ &-\tfrac{1}{3}(T^2(z) + T^2(z'))\partial_z\delta(z-z') - \tfrac{5}{72}(w_1^2(z) + w_1^2(z'))\partial_z^3\delta(z-z')\\ &-\tfrac{1}{108}(w_1^4(z) + w_1^4(z'))\partial_z\delta(z-z') + \tfrac{1}{24}(w_1^{2''}(z) + w_1^{2''}(z'))\partial_z\delta(z-z')\\ &-\tfrac{1}{9}(T(z)w_1^2(z) + T(z')w_1^2(z'))\partial_z\delta(z-z').\end{aligned} \tag{23}$$

(This is easily checked to be consistent with the Jacobi identity $\{w_1, \{w_3, w_3\}\} = 0$.)

The commutation relations (15)-(17) together with (21)-(23) for $b = \frac{2}{27}$ form a spin-3 operator algebra which fails to be a Lie algebra due to the non-linearity of its determining relation for the spin-3 commutator. However, the Jacobi identity is satisfied for all three generators of the algebra; in the present framework this is always guaranteed by the defining properties of the Gelfand-Dickey algebras [8, 9]. Note that for $w_1(z) = 0$ the commutation relations above reduce to Zamolodchikov's spin-3 algebra generated by the stress-energy tensor T and a chiral conformal field of weight 3 [1-4].

More generally, we can form spin-n extended conformal algebras generated by chiral conformal fields of weight $1, 2, \ldots, n$. Setting the U(1) currents $u_{n-1}(z)$ to zero we obtain the spin-n algebras W_n considered in [1-4, 7]. From the point of view of Gelfand-Dickey algebras the elimination of the U(1) current $u_{n-1}(z)$ is equivalent to considering the algebra GD(sl(n)) which is defined with respect to the operators $\partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0$ in analogy with the GD(gl(n)) case [3]. Here $X_f = \sum_{i=1}^n \partial^{-i}x_i$ with $x_i = \delta f/\delta u_{i-1}$ for $i = 1, 2, \ldots, n-1$ and the x_n is determined by the relation

$$\operatorname{res}[L_n, X_f] = 0. \tag{24}$$

The last equation is required for consistency of the choice $u_{n-1} = 0$ [8]. This is also consistent with the fact that the space of differential operators L_n with $u_{n-1} = 0$ is invariant under the action of Diff S^1 provided that the $\partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0$ are taken to act on densities of weight $-(n-1)/2$.

Finally, we comment on the description of non-Abelian Kac-Moody symmetries (and other extended symmetries) using the theory of Gelfand-Dickey algebras. For this purpose we consider differential operators L_n which are valued in a Lie algebra $\mathcal{G}$; the bracket $\{\,,\}_n$ can be taken as

$$\{f, g\}_n = \int \operatorname{Tr} \operatorname{res}(V_{X_f}(L_n)X_g) \tag{25}$$

where Tr denotes the trace for some particular matrix representation of $\mathcal{G}$. The simplest example is provided by $L_1 = \mathbb{1}\partial + u^a(z)T^a$ where $\mathbb{1}$ denotes the identity matrix and T^a are the generators of a compact simple Lie algebra $\mathcal{G}$ with commutation relations $[T^a, T^b] = C_d^{ab}T^d$. Assuming the normalisation $\operatorname{Tr}(T^aT^b) = \kappa\delta_{a,b}$ we find that the bracket (25) gives

$$\{u^a(z), u^b(z')\}_1 = C_d^{ab}(u^d(z) + u^d(z'))\delta(z-z') + \kappa\delta_{a,b}\partial_z\delta(z-z') \tag{26}$$

which is recognised as the centrally extended $\mathcal{G}$ Kac-Moody algebra. Application of equation (25) to higher-order matrix-valued operators L_n would yield extended (conformal) symmetries that contain non-Abelian Kac-Moody generators. The detailed investigation of these algebras will be presented elsewhere.

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Appendix 1

$$\begin{aligned}
\{f, g\}_2 = \int \mathrm{d}\tilde{z}\, &\frac{\delta g}{\delta u_1(\tilde{z})}\left[-\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)'' + 2\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)' + \left(u_1(\tilde{z})\frac{\delta f}{\delta u_0(\tilde{z})}\right)'\right] \\
&+ \frac{\delta g}{\delta u_0(\tilde{z})}\left[-\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)''' + \left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)'' - \left(u_0(\tilde{z})\frac{\delta f}{\delta u_0(\tilde{z})}\right)' - u_0(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)'\right. \\
&+ \left(u_1'(\tilde{z})\frac{\delta f}{\delta u_0(\tilde{z})}\right)' + u_1'(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)' + u_1(\tilde{z})\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)' \\
&\left. + u_1(\tilde{z})\left(u_1(\tilde{z})\frac{\delta f}{\delta u_0(\tilde{z})}\right)'\right].
\end{aligned}$$

Appendix 2

$$
\begin{aligned}
\{f, g\}_3 = \int \mathrm{d}\tilde{z} \frac{\delta g}{\delta u_2(\tilde{z})} \Bigg[& \left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)''' - 3\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)'' + 3\left(\frac{\delta f}{\delta u_2(\tilde{z})}\right)' + \left(u_1(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)' \\
& - \left(u_2(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)'' + 2\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_1(\tilde{z})}\right)' \Bigg] \\
& + \frac{\delta g}{\delta u_1(\tilde{z})} \Bigg[2\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)^{(4)} - 5\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)''' + 3\left(\frac{\delta f}{\delta u_2(\tilde{z})}\right)'' \\
& + 2\left(u_1(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)'' - u_1(\tilde{z})\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)' - u_0(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)' \\
& - \left(u_1(\tilde{z}) \frac{\delta f}{\delta u_1(\tilde{z})}\right)' - 2\left(u_0(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)' - 2\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)''' \\
& + u_2(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)''' + 3\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_1(\tilde{z})}\right)'' - 3u_2(\tilde{z})\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)'' \\
& + 2u_2(\tilde{z})\left(\frac{\delta f}{\delta u_2(\tilde{z})}\right)' - u_2(\tilde{z})\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)'' \\
& + 2u_2(\tilde{z})\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_1(\tilde{z})}\right)' + u_2(\tilde{z})\left(u_1(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)' \Bigg] \\
& + \frac{\delta g}{\delta u_0(\tilde{z})} \Bigg[\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)^{(5)} - 2\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)^{(4)} + \left(\frac{\delta f}{\delta u_2(\tilde{z})}\right)''' \\
& + \left(u_1(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)''' + u_1(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)''' - 2u_0(\tilde{z})\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)' \\
& + u_0(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)'' + u_1(\tilde{z})\left(u_1(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)' + u_1(\tilde{z})\left(\frac{\delta f}{\delta u_2(\tilde{z})}\right)' \\
& - \left(u_0(\tilde{z}) \frac{\delta f}{\delta u_1(\tilde{z})}\right)' - 2u_1(\tilde{z})\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)'' - \left(u_0(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)'' \\
& + u_2(\tilde{z})\left(\frac{\delta f}{\delta u_0(\tilde{z})}\right)^{(4)} - \left(u_2(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)^{(4)} - 2u_2(\tilde{z})\left(\frac{\delta f}{\delta u_1(\tilde{z})}\right)''' \\
& + \left(u_2(\tilde{z}) \frac{\delta f}{\delta u_1(\tilde{z})}\right)''' + u_2(\tilde{z})\left(\frac{\delta f}{\delta u_2(\tilde{z})}\right)'' - u_2(\tilde{z})\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)''' \\
& + u_2(\tilde{z})\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_1(\tilde{z})}\right)'' + u_2(\tilde{z})\left(u_1(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)'' \\
& - u_1(\tilde{z})\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)'' + u_1(\tilde{z})\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_1(\tilde{z})}\right)' \\
& - u_0(\tilde{z})\left(u_2(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)' - u_2(\tilde{z})\left(u_0(\tilde{z}) \frac{\delta f}{\delta u_0(\tilde{z})}\right)' \Bigg].
\end{aligned}
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